

Spectral Radius and Degree Sequence of a Graph

Chia-an Liu^{* †} Chih-wen Weng[†]

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Abstract

Let G be a simple connected graph of order n with degree sequence d_1, d_2, \dots, d_n in non-increasing order. The *spectral radius* $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix. For each positive integer ℓ at most n , we give a sharp upper bound for $\rho(G)$ by a function of d_1, d_2, \dots, d_ℓ , which generalizes a series of previous results.

Keywords: Graph, adjacency matrix, spectral radius, degree sequence.

1 Introduction

Let G be a simple connected graph of n vertices and m edges with degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$. The *adjacency matrix* $A = (a_{ij})$ of G is a binary square matrix of order n with rows and columns indexed by the vertex set VG of G such that for any $i, j \in VG$, $a_{ij} = 1$ if i, j are adjacent in G . The *spectral radius* $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix, which has been studied by many authors.

The following theorem is well-known [1, Chapter 2].

^{*}Corresponding author. E-mail address: twister.imm96g@g2.nctu.edu.tw (C.-A Liu).

[†]Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C..

Theorem 1.1. *If A is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(A)$ and row-sums r_1, r_2, \dots, r_n , then*

$$\rho(A) \leq \max_{1 \leq i \leq n} r_i$$

with equality if and only if the row-sums of A are all equal.

In 1985 [2, Corollary 2.3], Brauldi and Hoffman showed the following result.

Theorem 1.2. *If $m \leq k(k-1)/2$, then*

$$\rho(G) \leq k-1$$

with equality if and only if G is isomorphic to the complete graph K_n of order n .

In 1987 [3], Stanley improved Theorem 1.2 and showed the following result.

Theorem 1.3.

$$\rho(G) \leq \frac{-1 + \sqrt{1 + 8m}}{2}$$

with equality if and only if G is isomorphic to the complete graph K_n of order n .

In 1998 [4, Theorem 2], Yuan Hong improved Theorem 1.3 and showed the following result.

Theorem 1.4.

$$\rho(G) \leq \sqrt{2m - n + 1}$$

with equality if and only if G is isomorphic to the star $K_{1,n-1}$ or to the complete graph K_n .

In 2001 [5, Theorem 2.3], Hong et al. improved Theorem 1.4 and showed the following result.

Theorem 1.5.

$$\rho(G) \leq \frac{d_n - 1 + \sqrt{(d_n + 1)^2 + 4(2m - nd_n)}}{2}$$

with equality if and only if G is regular or there exists $2 \leq t \leq n$ such that $d_1 = d_{t-1} = n - 1$ and $d_t = d_n$.

In 2004 [6, Theorem 2.2], Jinlong Shu and Yarong Wu improved Theorem 1.1 in the case that A is the adjacency matrix of G by showing the following result.

Theorem 1.6. For $1 \leq \ell \leq n$,

$$\rho(G) \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}$$

with equality if and only if G is regular or there exists $2 \leq t \leq \ell$ such that $d_1 = d_{t-1} = n - 1$ and $d_t = d_n$.

Moreover, they also showed in [6, Theorem 2.5] that if $p + q \geq d_1 + 1$ then Theorem 1.6 improves Theorem 1.5 where p is the number of vertices with the largest degree d_1 and q is the number of vertices with the second largest degree. The special case $\ell = 2$ of Theorem 1.6 is reproved [7].

In this research, we present a sharp upper bound of $\rho(G)$ in terms of the degree sequence of G , which improves all the above theorems.

Theorem 1.7. For $1 \leq \ell \leq n$,

$$\rho(G) \leq \phi_\ell := \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4 \sum_{i=1}^{\ell-1} (d_i - d_\ell)}}{2},$$

with equality if and only if G is regular or there exists $2 \leq t \leq \ell$ such that $d_1 = d_{t-1} = n - 1$ and $d_t = d_n$.

This result improves Theorem 1.5 and Theorem 1.6 since ϕ_n is exactly the upper bounds in Theorem 1.5 and is at most the upper bound appearing in Theorem 1.6.

Note that the number ϕ_ℓ defined in Theorem 1.7 is at least d_ℓ . The sequence $\phi_1, \phi_2, \dots, \phi_n$ is not necessary to be non-increasing. We show that this sequence is first non-increasing and then non-decreasing, and determine its lowest value in Section 3.

2 Proof of Theorem 1.7

Proof. Let the vertices be labeled by $1, 2, \dots, n$ with degrees $d_1 \geq d_2 \geq \dots \geq d_n$, respectively. For each $1 \leq i \leq \ell - 1$, let $x_i \geq 1$ be a variable to be determined later. Let $U = \text{diag}(x_1, x_2, \dots, x_{\ell-1}, 1, 1, \dots, 1)$ be a diagonal matrix of size $n \times n$. Then $U^{-1} = \text{diag}(x_1^{-1}, x_2^{-1}, \dots, x_{\ell-1}^{-1}, 1, 1, \dots, 1)$. Let $B = U^{-1}AU$. Note that A and B have the same eigenvalues. Let r_1, r_2, \dots, r_n be the row-sums of B . Then for $1 \leq i \leq \ell - 1$ we have

$$\begin{aligned} r_i &= \sum_{k=1}^{\ell-1} \frac{x_k}{x_i} a_{ik} + \sum_{k=\ell}^n \frac{1}{x_i} a_{ik} = \frac{1}{x_i} \sum_{k=1}^n a_{ik} + \frac{1}{x_i} \sum_{k=1}^{\ell-1} (x_k - 1) a_{ik} \\ &\leq \frac{1}{x_i} d_i + \frac{1}{x_i} \left(\sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right), \end{aligned} \quad (2.1)$$

and for $\ell \leq j \leq n$ we have

$$\begin{aligned} r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk} + \sum_{k=\ell}^n a_{jk} = \sum_{k=1}^n a_{jk} + \sum_{k=1}^{\ell-1} (x_k - 1) a_{jk} \\ &\leq d_j + \left(\sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right). \end{aligned} \quad (2.2)$$

For $1 \leq i \leq \ell - 1$ let

$$x_i = 1 + \frac{d_i - d_\ell}{\phi_\ell + 1} \geq 1, \quad (2.3)$$

where ϕ_ℓ is defined in Theorem 1.7. Then for $1 \leq i \leq \ell - 1$ we have

$$r_i \leq \frac{1}{x_i} d_i + \frac{1}{x_i} \left(\sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right) = \phi_\ell,$$

and for $\ell \leq j \leq n$ we have

$$r_j \leq d_j + \left(\sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) = \phi_\ell.$$

Hence by Theorem 1.1,

$$\rho(G) = \rho(B) \leq \max_{1 \leq i \leq n} \{r_i\} \leq \phi_\ell. \quad (2.4)$$

The first part of Theorem 1.7 follows.

The sufficient condition of $\phi_\ell = \rho(G)$ follows from the fact that

$$\phi_\ell \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}$$

and applying the second part in Theorem 1.6.

To prove the necessary condition of $\phi_\ell = \rho(G)$, suppose $\phi_\ell = \rho(G)$. Then the equalities in (2.1) and (2.2) all holds. If $d_1 = d_\ell$, then $d_1 = \phi_1 = \phi_\ell = \rho(G)$, and G is regular by the second part of Theorem 1.1. Suppose $2 \leq t \leq \ell$ such that $d_{t-1} > d_t = d_\ell$. Then $x_i > 1$ for $1 \leq i \leq t-1$ by (2.3). For each $1 \leq i \leq \ell-1$, the equality in (2.1) implies that $a_{ik} = 1$ for $1 \leq k \leq t-1$, $k \neq i$. For each $\ell \leq j \leq n$, the equality in (2.2) implies that $a_{jk} = 1$ for $1 \leq k \leq t-1$ and $d_j = d_\ell$. Hence $n-1 = d_1 = d_{t-1} > d_t = d_\ell = d_n$.

We complete the proof. \square

3 The sequence $\phi_1, \phi_2, \dots, \phi_n$

The sequence $\phi_1, \phi_2, \dots, \phi_n$ is not necessarily non-increasing. For example, the path P_n of n vertices has $2 = d_1 = d_{n-2} > d_{n-1} = d_n = 1$, and it is immediate to check that if $n \geq 6$ then $\phi_1 = \phi_2 = 2 < \sqrt{n-1} = \phi_{n-1} = \phi_n$.

Clearly that for all $1 \leq s < t \leq n$, $d_s = d_t$ implies that $\phi_s = \phi_t$. However, $\phi_s = \phi_t$ dose not imply $d_s = d_t$. For example, in the graph with degree sequence $(4, 3, 3, 2, 1, 1)$, one can check that $\phi_4 = \phi_5 = 3$ but $d_4 > d_5$.

Recall that $d_s = d_{s+1}$ implies $\phi_s = \phi_{s+1}$ for $1 \leq s \leq n-1$. The following proposition describes the shape of the sequence $\phi_1, \phi_2, \dots, \phi_n$.

Proposition 3.1. *Suppose $d_s > d_{s+1}$ for $1 \leq s \leq n-1$, and let $\succeq \in \{>, =\}$. Then*

$$\phi_s \succeq \phi_{s+1} \quad \text{iff} \quad \sum_{i=1}^s d_i \succeq s(s-1).$$

Proof. Recall that

$$\phi_s = \frac{d_s - 1 + \sqrt{(d_s + 1)^2 + 4 \sum_{i=1}^{s-1} (d_i - d_s)}}{2}.$$

Consider the following equivalent relations step by step.

$$\begin{aligned}
& \phi_s \succeq \phi_{s+1} \\
\Leftrightarrow & \quad d_s - d_{s+1} + \sqrt{(d_s + 1)^2 + 4 \sum_{i=1}^{s-1} (d_i - d_s)} \\
& \quad \succeq \sqrt{(d_{s+1} + 1)^2 + 4 \sum_{i=1}^s (d_i - d_{s+1})} \\
\Leftrightarrow & \quad \sqrt{(d_s + 1)^2 + 4 \sum_{i=1}^{s-1} (d_i - d_s)} \succeq 2s - (d_s + 1) \\
\Leftrightarrow & \quad (d_s + 1)^2 + 4 \sum_{i=1}^s (d_i - d_s) \succeq 4s^2 - 4s(d_s + 1) + (d_s + 1)^2 \\
\Leftrightarrow & \quad \sum_{i=1}^s d_i \succeq s(s - 1),
\end{aligned}$$

where the third relation is obtained from the second by taking square on both sides, simplifying it, and deleting the common term $d_s - d_{s+1}$. \square

Corollary 3.2. *Let $3 \leq \ell \leq n$ be the smallest integer such that $\sum_{i=1}^{\ell} d_i < \ell(\ell - 1)$. Then for $1 \leq j \leq n$ we have*

$$\phi_j = \min\{\phi_k \mid 1 \leq k \leq n\}$$

if and only if $d_j = d_{\ell}$, or $d_j = d_{\ell-1}$ with $\sum_{i=1}^{\ell-1} d_i = (\ell - 1)(\ell - 2)$.

Proof. From Proposition 3.1, $\sum_{i=1}^{\ell-1} d_i = (\ell - 1)(\ell - 2)$ implies $\phi_{\ell-1} = \phi_{\ell}$. Also, clearly that $d_j = d_{\ell}$ implies $\phi_j = \phi_{\ell}$. We show that $\phi_{\ell} = \min\{\phi_k \mid 1 \leq k \leq n\}$ in the following.

For $1 \leq s \leq \ell - 1$, from Proposition 3.1 we have $\phi_s \geq \phi_{s+1}$ since $\sum_{i=1}^s d_i \geq s(s - 1)$. For $\ell \leq t \leq n - 1$, note that $\sum_{i=1}^t d_i < t(t - 1)$ implies $d_t < t - 1$, and hence $\sum_{i=1}^{t+1} d_i < t(t - 1) + (t - 1) < t(t + 1)$. From Proposition 3.1 we have $\phi_{\ell} \leq \phi_{\ell+1} \leq \dots \leq \phi_n$ since $\sum_{i=1}^{\ell} d_i < \ell(\ell - 1)$. The result follows. \square

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Chia-an Liu
Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsueh Road
Hsinchu, Taiwan 300, R.O.C.
Email: twister.imm96g@g2.nctu.edu.tw
Ext: +886-3-5712121-56460